Logical relations for call-by-push-value models, via internal fibrations in a 2-category

***Abstract*—We give a denotational account of logical relations for call-by-push-value (CBPV) in the fibrational style of Hermida, Jacobs, Katsumata and others. Fibrations—which axiomatise the usual notion of sets-with-relations—provide a clean framework for constructing new, logical relations-style, models. Such models can then be used to study properties such as effect simulation.**

Extending this picture to CBPV is challenging: the models incorporate both adjunctions and enrichment, making the appropriate notion of fibration unclear. We handle this using 2-category theory: we identify an appropriate 2-category, and define CBPV fibrations to be fibrations internal to this 2-category which strictly preserve the CBPV semantics.

Next, we develop the theory so it parallels the classical setting. We give versions of the codomain and subobject fibrations, and show that new models can be constructed from old ones by pullback. The resulting framework enables the construction of new, logical relations-style, models for CBPV.

Finally, we demonstrate the utility of our approach with particular examples. These include a generalisation of Katsumata's $\top\top$-lifting to CBPV models, an effect simulation result, and a novel relative full completeness result for CBPV.

I. Introduction

Logical relations are a fundamental tool for proving metatheoretic properties of logics and programming languages: for a flavour of their longevity and range of application, see [1], [2], [3], [4]. In their simplest form, logical relations are families of relations over closed terms, defined by induction on the types. Denotationally, this data typically organises itself into a *relations model* (e.g. [53], [2], [61]). A priori there are many different forms of relations models. However, Jacobs [6] and Hermida [5] have shown that they can studied in general using *Grothendieck fibrations*, a category-theoretic abstraction which axiomatises the notion of relation. Because of their rich mathematical theory, Grothendieck fibrations provide a robust framework for constructing new models from old ones, as well as capturing many existing constructions.

In recent years there has been extensive work extending fibrational techniques to semantic accounts of logical relations in the presence of effects, in particular monadic models of call-by-value (CBV) languages (e.g. [41], [40], [8], [42], [7]). This theory forms a powerful and flexible framework for building semantic models, which has been used for attacking problems such as definability [9], [10] and effect simulation [16]. However, a corresponding framework is currently lacking for models of call-by-push-value (CBPV) in their full generality. While particular cases have been studied (e.g. [50], [54]), we lack a denotational account of logical relations of CBPV that matches the generality and flexibility of existing work on CBV models.

This paper aims to resolve the gap. We introduce a mathematically-justified notion of fibrations for CBPV models (Section IV), and develop their basic theory. Then we prove a *lifting theorem* (Section V) which yields a modular framework for building new CBPV models from existing ones, as we show through a range of examples (Section V-A). In particular, we see that our definition specialises to the expected syntactic relation on terms. We then put this theory to work by showing how to prove effect simulation results analogously to Katsumata's argument [16] for CBV models (Section VI). As a consequence of our foundational perspective, moreover, we are able to prove a relative full completeness result (Section VII) establishing semantically that CBPV function types conservatively extend the first-order fragment.

Fibrations for semantic models. To explain what universal property we are looking for, and the obstacles to getting it, let us first consider the situations for the pure and CBV cases.

In semantic terms, Grothendieck fibrations axiomatise the following situation. Let $\mathbf{Pred}$ be the category in which objects are sets X equipped with a subset (predicate) $R \subseteq X$, and morphisms $(X, R) \to (X', R')$ are functions $f : X \to X'$ preserving the predicates: if $x \in R$ then $f(x) \in R'$. There is a forgetful functor $p : \mathbf{Pred} \to \mathbf{Set}$ with the property that, if $f : X \to p(Y, S)$ in $\mathbf{Set}$, then there is a canonical way to lift f to a morphism in $\mathbf{Pred}$, using the inverse image: $f : (X, f^{-1}[S]) \to (Y, S)$. Thus, a fibration $p : \mathbb{E} \to \mathbb{B}$ may be thought of saying objects of $\mathbb{E}$ are "relations" on $\mathbb{B}$-objects. Furthermore, $\mathbf{Pred}$ is cartesian closed and the forgetful functor $p : \mathbf{Pred} \to \mathbf{Set}$ preserves this structure on the nose. Thinking of these categories as models of the simply-typed λ-calculus (STLC), therefore, we see that (1) a λ-term is interpreted as a morphism preserving a certain predicate; (2) the functor p preserves the interpretation of λ-terms. Thus, the model in $\mathbf{Pred}$ refines that in $\mathbf{Set}$ with extra information encoded by the relations.

A key feature of this approach is that it is easy to construct new fibrational models from old ones. Consider, for instance, the pullback square on the left below, where $\Delta(X) := X \times X$:

$$\begin{array}{ccc} \mathbf{BinPred} & \longrightarrow & \mathbf{Pred} \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathbf{Set} & \xrightarrow{\Delta} & \mathbf{Set} \end{array} \qquad \begin{array}{ccc} \mathbb{P} & \longrightarrow & \mathbb{E} \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array} \tag{1}$$

The objects in $\mathbf{BinPred}$ are sets X with a binary relation

$R \subseteq X \times X$, and the maps are functions which preserve the relation. $\mathbf{BinPred}$ is still cartesian closed and the forgetful functor q is still a fibration so, from our initial fibration, we have constructed a new model.

This holds in general. Following Katsumata [16], let us call a fibration which strictly preserves cartesian closed structure a *STLC fibration*. For any STLC fibration p as on the right above, the pullback q along a product-preserving functor F will also be an STLC fibration. Thus, starting from STLC fibrations we have a robust and flexible way to construct a wide range of STLC models.

Lifting CBV models. The framework just outlined for constructing new models extends to the effectful setting. Let us call a *(Moggi-style) CBV model* a cartesian closed category $\mathbb{C}$ equipped with a strong monad (T, μ, η, t) (see e.g. [14], [15]). An effectful program $\Gamma \vdash M : A$ is then interpreted as a morphism $[\![\Gamma]\!] \to T[\![A]\!]$ in $\mathbb{C}$ [12], [13]. We want a "CBV fibration" to capture situations when one CBV model is a refinement of another, just as we had for STLC fibrations. Thus, the appropriate notion should be a fibration which strictly preserves the interpretation of terms. If $(\widehat{\mathbb{C}}, \widehat{T})$ and $(\mathbb{C}, T)$ are CBV models, we say $p : \widehat{\mathbb{C}} \to \mathbb{C}$ is a *CBV fibration* if is an STLC fibration which also preserves all the monadic structure, so that $p\widehat{T} = Tp$, $p\mu^{\widehat{T}}_{C} = \mu^{T}_{pC}$, $p\eta^{\widehat{T}}_{C} = \eta^{T}_{p}C$, $p\eta^{\widehat{T}}_{C} = \eta^{T}_{pC}$ and $pt^{\widehat{T}}_{C} = t^{T}_{pC}$ for every $C \in \mathbb{C}$.

Just as every STLC fibration led to new STLC models, so every CBV fibration leads to new CBV models, as follows. Recall that if (S, t^S) is a strong monad on $\mathbb{B}$ and (T, t^T) is a strong monad on $\mathbb{C}$ then a *strong monad morphism* consists of a functor $F : \mathbb{B} \to \mathbb{C}$ and a natural transformation $\gamma : FS \Rightarrow TF$ which is compatible with the unit, multiplication, and strengths (see e.g. [17], [16]). Given a CBV fibration p and a strong monad morphism (F, γ) in which F is cartesian, there is a universal choice of model $(\widehat{\mathbb{B}}, \widehat{S})$ lying over $(\mathbb{B}, S)$:

$$\begin{array}{ccc} (\widehat{\mathbb{B}}, \widehat{S}) & \dashrightarrow & (\widehat{\mathbb{C}}, \widehat{T}) \\ \downarrow & & \downarrow{\scriptstyle p} \\ (\mathbb{B}, S) & \xrightarrow[(F,\gamma)]{} & (\mathbb{C}, T) \end{array} \tag{2}$$

The monad $\widehat{S}$ is defined at an object $(\mathcal{B} \in \mathbb{B}, \widehat{C} \in \widehat{\mathbb{C}})$ by applying the universal property of the fibration p to the arrow

$$\gamma_B : FS(B) \to TF(B) = p(\widehat{T}\widehat{C})$$

We can also use the language of fibrations to express this universal property, as follows.

Definition I.1. The category $\mathbf{CBV}^{\times}$ has objects CBV models $(\mathbb{C}, T)$ and morphisms strong monad morphisms (F, γ) in which F is a cartesian functor. Note that F need not preserve exponentials. The category $\mathbf{CBV}_l$ of *CBV model liftings* has objects CBV fibrations $p : (\widehat{\mathbb{C}}, \widehat{T}) \to (\mathbb{C}, T)$. A morphism $p \to p'$ consists of $\mathbf{CBV}^{\times}$-maps $(\widehat{F}, \widehat{\gamma}) : (\widehat{\mathbb{C}}, \widehat{T}) \to (\widehat{\mathbb{C}'}, \widehat{T'})$ and $(F, \gamma) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ which commute with the fibrations, so that $p' \circ \widehat{F} = F \circ p$, and similarly on the natural transformations γ and $\widehat{\gamma}$.

We then obtain the following *lifting result* (cf. [8], [10]), which is the CBV-model version of the pullback in (1).

Proposition I.2 (Generalised $\top\top$-lifting)**.** *The functor* $\mathbf{CBV}_l \to \mathbf{CBV}^{\times}$ *sending a CBV fibration* p *to its codomain is a fibration.*

From CBV to CBPV. We want a version of the theory just sketched, but for CBPV. Thus, we require (1) a notion of CBPV fibration, and (2) a lifting theorem as in Proposition I.2. However, a CBPV model consists of a cartesian category $\mathbb{C}$ and a *locally* $\mathbb{C}$*-indexed category* $\mathcal{C}$ (see Definition III.1); abstractly, a locally $\mathbb{C}$-indexed category is a category enriched in the presheaf category $\mathcal{P}\mathbb{C}$.

It follows that we cannot ask CBPV fibrations to be functors preserving the semantics: this is not even well-typed. Moreover, just as in (2) we allowed the base categories to vary, morphisms of CBPV models may also change the enriching category $\mathbb{C}$. Thus, we cannot even define the right notion of fibration by taking the $\mathcal{P}\mathbb{C}$-enriched definition (and if we could, the theory of enriched fibrations [18], [19], [20] is motivated by quite different concerns—namely the correspondence with the Grothendieck construction—so it is not clear this would do the right thing). Nor is it straightforward to simply write a definition by hand: if one looks at the concrete definition of morphisms of CBPV models, there appear to be choices in how to define the universal property of a "CBPV fibration" (see Remark IV.7). To obtain a principled definition of CBPV fibration and its corresponding theory, therefore, we must look elsewhere. This is the technical core of the paper.

Outline of the paper. We use 2-category theory to resolve the difficulties in defining CBPV fibrations. We introduce a 2-category $\mathbf{LInd}$ of locally indexed categories (Section IV-A) and identify the fibrations internal to this 2-category (Section IV-B). Just as an STLC fibration is a fibration in $\mathbf{Cat}$ which preserves STLC model structure, so we define a CBPV fibration to be a fibration in $\mathbf{LInd}$ which preserves CBPV model structure (Section IV-D). An immediate consequence of this approach is that the rich theory of fibrations still applies to our definition. We highlight this by developing locally indexed versions of important constructs in the theory of STLC and CBV fibrations, such as the codomain and subobject fibrations.

Next, we prove a lifting theorem for CBPV fibrations (Theorem V.2), thereby regaining the situation sketched in Proposition I.2. As for CBV fibrations and STLC fibrations, this is a useful source of examples: we sketch several in Section V-A, then show how our theory can be used to adapt Katsumata's effect simulation framework [16] from CBV to CBPV (Section VI).

The foundational nature of our theory means that we have many of the ingredients needed to prove a semantic relative full completeness result in the style of Lafont [56]. We do this in Section VII, thereby obtaining via our semantic technology a proof that CBPV function types are conservative over the first-order fragment. Finally, in Section VIII we outline the sense in which our lifting theorem is part of a general mathematical phenomenon, which applies both to CBV models and to CBPV models.

Notation. We assume familiarity with Grothendieck fibrations: for an introduction, see e.g. [6]. We write $\mathbb{C}^{\to}$ for the arrow category, in which objects are maps in $\mathbb{C}$ and morphisms are commuting squares, and $\mathrm{Sub}\,\mathbb{C}$ for the full subcategory obtained by restricting the objects to monos. In both cases we write cod for the codomain functor into $\mathbb{C}$. If $p : \mathbb{E} \to \mathbb{B}$ is a fibration, we denote the products and exponentials in $\mathbb{E}$ by $*$ and $\supset$. We assume throughout that all fibrations are split.

II. 2-Category Theory

We assume the basics of 2-category theory, in particular the definition of 2-categories, 2-functors, and transformations. For a textbook length introduction, see e.g. [21], [22]. We also provide a brief summary in Appendix A.

We write $\mathbb{C}, \mathbb{D}, \ldots$ for categories and $\mathcal{C}, \mathcal{D}, \ldots$ for 2-categories. The four 2-categories of 2-functors $\mathcal{C} \to \mathcal{D}$, strict / pseudo / lax / oplax natural transformations, and modifications, are denoted $[\mathcal{C}, \mathcal{D}]_{\mathrm{st}}, [\mathcal{C}, \mathcal{D}]_{\mathrm{ps}}, [\mathcal{C}, \mathcal{D}]_{\mathrm{lx}}$ and $[\mathcal{C}, \mathcal{D}]_{\mathrm{oplx}}$, respectively. Lax natural transformations are directed as follows:

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FC' \\ {\scriptstyle \sigma_C}\downarrow & \overset{\sigma_f}{\swarrow} & \downarrow{\scriptstyle \sigma_{C'}} \\ GC & \xrightarrow[Gf]{} & GC' \end{array}$$

We use the following notation for common 2-categories. Recall that a cartesian category is *distributive* if it has finite coproducts and the canonical morphism $[X \times \mathrm{id}_B, X \times \mathrm{id}_C] : (X \times B) + (X \times C) \to X \times (B + C)$ is invertible, and that a *bicartesian closed category* (or *biCCC*) is a cartesian closed category with finite coproducts.

- $\mathbf{Cat}$ is the 2-category of categories.
- $\mathbf{CartCat}$ and $\mathbf{DistCat}$ are the 2-categories of cartesian categories and distributive categories, respectively. In each case the 1-cells are functors preserving the structure up to isomorphism, and the 2-cells are all natural transformations. We write $\mathbf{CartCat}_{\mathrm{st}}$ and $\mathbf{DistCat}_{\mathrm{st}}$ for the sub-2-categories with the same objects and functors strictly preserving the structure.

A. Adjunctions and their morphisms

CBPV models are defined using adjunctions internal to a 2-category. We recall the definition.

Definition II.1. An *adjunction* in a 2-category $\mathcal{C}$ consists of x1-cells $f : A \leftrightarrows B : u$ together with 2-cells $\eta : \mathrm{id}_A \Rightarrow u \circ f$ and $\varepsilon : f \circ u \Rightarrow \mathrm{id}_B$ satisfying the usual triangle laws.

Example II.2. An adjunction in $\mathbf{CartCat}$ is an adjunction between cartesian categories such that both the left and right adjoints are cartesian.

We shall also need morphisms between adjunctions. For this we shall see adjunctions as certain 2-functors and then define maps of adjunctions and their 2-cells as the corresponding transformations and modifications (cf. [23], [24]).

Let $\mathbf{Adj}$ be the 2-category freely generated by the data of an adjunction, namely two objects $\bullet$ and $*$, 1-cells $f : \bullet \leftrightarrows * : u$, and 2-cells $\eta : \mathrm{id}_\bullet \Rightarrow u \circ f$ and $\varepsilon : f \circ u \Rightarrow \mathrm{id}_*$ satisfying the triangle laws. A 2-functor $\mathbf{Adj} \to \mathcal{C}$ is then equivalently an adjunction in $\mathcal{C}$. It follows immediately that any 2-functor preserves adjunctions.

Definition II.3. We write $\mathrm{Adj}(\mathcal{C})_{\mathrm{w}}$ for the 2-functor category $[\mathbf{Adj}, \mathcal{C}]_{\mathrm{w}}$, where $\mathrm{w} \in \{\mathrm{st}, \mathrm{ps}, \mathrm{lx}, \mathrm{oplx}\}$. We call the 1-cells *strict / pseudo / lax / oplax adjunction maps* and the 2-cells *adjunction 2-cells.*

A lax adjunction map $(\ell : X \leftrightarrows Y : r) \to (f : A \leftrightarrows B : u)$ consists of 1-cells $m : X \to A$ and $n : Y \to B$ together with 2-cells as shown below

$$\begin{array}{ccc} X & \xrightarrow{\ell} & Y \\ {\scriptstyle m}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle n} \\ A & \xrightarrow[f]{} & B \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{r} & X \\ {\scriptstyle n}\downarrow & \overset{\beta}{\swarrow} & \downarrow{\scriptstyle m} \\ B & \xrightarrow[u]{} & A \end{array}$$

which are moreover compatible with the units and counits, in the sense that the following two diagrams commute:

$$\begin{array}{ccc} n\ell r & \xrightarrow{n\varepsilon^{\ell,n}} & n \\ {\scriptstyle \alpha r}\downarrow & & \uparrow{\scriptstyle \varepsilon^{f,u}n} \\ fmr & \xrightarrow[f\beta]{} & fun \end{array} \qquad\qquad \begin{array}{ccc} \ell & \xrightarrow{\ell\eta^{r,\ell}} & mr\ell \\ {\scriptstyle \eta^{u,f}\ell}\downarrow & & \downarrow{\scriptstyle \beta\ell} \\ uf\ell & \xleftarrow[n\alpha]{} & un\ell \end{array} \tag{3}$$

This is a strict adjunction map when α and β are both the identity. One then recovers the notion of morphism of adjunctions commonly used for showing the terminality of the Eilenberg–Moore category (as in e.g. [25, §IV.7]).

B. Fibrations

We shall make extensive use of fibrations internal to a 2-category. These have been studied extensively (e.g. [26], [27]); for a readable introduction to the theory, see [28].

Definition II.4. Let $\mathcal{C}$ be a 2-category. A *fibration* in $\mathcal{C}$ is a 1-cell $p : E \to B$ such that

1) For every $X \in \mathcal{C}$, the functor $p \circ (-) : \mathcal{C}(X, E) \to \mathcal{C}(X, B)$ is a fibration in $\mathbf{Cat}$, and
2) For every $h : Y \to X$ the following defines a morphism of fibrations (see e.g. [29, Definition 2.6]):

$$\begin{array}{ccc} \mathcal{C}(X,E) & \xrightarrow{(-)\circ h} & \mathcal{C}(Y,E) \\ {\scriptstyle p\circ(-)}\downarrow & & \downarrow{\scriptstyle p\circ(-)} \\ \mathcal{C}(X,B) & \xrightarrow[(-)\circ h]{} & \mathcal{C}(Y,B) \end{array}$$

An *opfibration* is (somewhat unfortunately) defined to be a fibration in $\mathcal{C}^{\mathrm{co}}$.

Fibrations in a 2-category inherit many of the properties of fibrations in $\mathbf{Cat}$. For example, it is immediate that the identity is always a fibration and that fibrations are closed under composition. An (op)fibration in $\mathbf{Cat}$ is exactly an (op)fibration in the usual sense.

The next result shows this is a general phenomenon about categories with algebraic structure. Just as algebras for monads describe algebraic structure on objects of a category, so algebras for 2-monads describe algebraic structure on categories. For an introduction to this approach, see [32]. For the definition of algebras and their morphisms, see e.g. [33], [34].

Proposition II.5. *1) If T is a 2-monad on a 2-category $\mathcal{C}$, and $(f, \overline{f})$ is a pseudomorphism or strict morphism of algebras T-pseudoalgebras such that f is a fibration in $\mathcal{C}$, then $(f, \overline{f})$ is a fibration in T-*Alg*.*
2) Right adjoint 2-functors preserve fibrations.
*Hence, $(f, \overline{f})$ is a fibration in T-*Alg* if and only if its underlying map is a fibration.*

This theorem covers $\mathbf{CartCat}$, $\mathbf{DistCat}$ and similar cases.

To characterise fibrations in our particular example, we will need some simple 2-categorical limits. For an extensive discussion of both limits and fibrations, see [30].

Definition II.6. Let $(A \xrightarrow{f} C \xleftarrow{g} B)$ be a cospan in a 2-category $\mathcal{C}$. The *comma object* $f \downarrow g$ is the universal object with a 2-cell as shown below: see e.g. [30, §2.1] for the precise universal property.

$$\begin{array}{ccc} f \downarrow g & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \overset{\Longrightarrow}{\scriptstyle \lambda} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

The *pullback* of g along f is defined analogously, except the square must be filled by an identity: see [30, §2.1].

It follows from the corresponding fact in $\mathbf{Cat}$ that fibrations in a 2-category are closed under pullbacks.

Example II.7. The comma object $(F \downarrow G)$ in $\mathbf{CartCat}$ is the usual comma category (e.g. [25, §II.6]) with cartesian structure $(FA \xrightarrow{j} GB) \times (FA' \xrightarrow{j'} GB')$ defined to be

$$F(A \times A') \xrightarrow{\cong} FA \times FA' \xrightarrow{j \times j'} GB \times GB' \xrightarrow{\cong} G(B \otimes B')$$

Similar remarks hold for $\mathbf{DistCat}$.

Example II.8. In any 2-category with comma objects the *arrow object* $C^{\rightarrow}$ on C is defined to be the comma object $(\mathrm{id}_C \downarrow \mathrm{id}_C)$. In $\mathbf{Cat}$ this is exactly the arrow category $\mathbb{C}^{\rightarrow}$. The definition of the comma object also gives a map we denote $\mathrm{cod} : C^{\rightarrow} \to C$; this is always an opfibration.

Example II.9. $\mathbf{DistCat}$ does not have all pullbacks. Indeed, recall that the pullback of $F : \mathbb{A} \to \mathbb{C}$ and $G : \mathbb{B} \to \mathbb{C}$ in $\mathbf{Cat}$ has objects pairs $(A \in \mathbb{A}, B \in \mathbb{B})$ such that $FA = GB$. But if F and G only preserve products up to isomorphism, $(A_1 \times A_2, B_1 \times B_2)$ may not be an object of the pullback even though both (A_i, B_i) are. However, $\mathbf{DistCat}_{\mathrm{st}}$ has all pullbacks. Moreover, if p is a bifibration which strictly preserves products and coproducts then the pullback along any map exists in $\mathbf{DistCat}$ (cf. [16, Proposition 6]). Similar remarks apply to $\mathbf{CartCat}$.

III. Denotational models of CBPV

We refer to Levy's extensive works [35], [36], [37] for the syntax and semantics of CBPV. There are several equivalent ways to phrase the data of a denotational model (see [38, Chapter 11] and [38, §15.1]), so we make our choice explicit.

A. Locally indexed categories

The basic data of a CBPV model is a *locally indexed adjunction*. This is an adjunction in the 2-category $\mathcal{P}(\mathbb{C})$-$\mathbf{Cat}$ of categories enriched in a presheaf category $\mathcal{P}\mathbb{C}$. We recall from [35], [36] some of the basic definitions. Throughout this section, fix $(\mathbb{C}, \times, 1)$ to be a cartesian category.

Definition III.1. A *locally $\mathbb{C}$-indexed category* $\mathcal{C}$ consists of:

1) A collection $|\mathbb{C}|$ of objects $A, B, \ldots$.
2) For each $c \in \mathbb{C}$ a category $\mathcal{C}_c$ with objects $|\mathbb{C}|$. Thus for each $A, B \in \mathcal{C}$ we have a set $\mathcal{C}_c(A, B)$ of *morphisms over c*, denoted $f : A \underset{c}{\to} B$;
3) For each $\rho : d \to c$ in $\mathbb{C}$ an identity-on-objects functor $(-) \triangleleft \rho : \mathcal{C}_c \to \mathcal{C}_d$, subject to the following axioms for $\rho' : e \to d$, $f : A \underset{c}{\to} B$ and $g : B \underset{c}{\to} C$:

$$f \triangleleft \mathrm{id}_c = f \qquad f \triangleleft (\rho \circ \rho') = (f \triangleleft \rho) \triangleleft \rho'$$
$$(g \circ f) \triangleleft \rho = (g \triangleleft \rho) \circ (f \triangleleft \rho)$$

Example III.2. The locally $\mathbb{C}$-indexed category $\mathsf{self}\,\mathbb{C}$ has objects as in $\mathbb{C}$ and $(\mathsf{self}\,\mathbb{C})_C(A, B) := \mathbb{C}(C \times A, B)$.

Definition III.3. A *locally $\mathbb{C}$-indexed functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a mapping $|F| : |\mathcal{C}| \to |\mathcal{D}|$ on objects and, for every $c \in \mathbb{C}$ and $A, B \in \mathcal{C}$, a mapping $F_c : \mathcal{C}_c(A, B) \to \mathcal{D}_c(FA, FB)$ such that F_c defines a functor $\mathcal{C}_c \to \mathcal{D}_c$ and is compatible with renaming: $F_c(f \triangleleft \rho) = F_c(f) \triangleleft \rho$ for any $\rho : d \to c$ in $\mathbb{C}$. We drop the subscripts where they are clear from context.

Definition III.4. A *locally $\mathbb{C}$-indexed transformation* $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ consists of an arrow $\alpha_C : FC \underset{1}{\to} GC$ for each $C \in \mathcal{C}$, natural in the sense that for any $f : B \underset{c}{\to} C$ in $\mathcal{C}$ the following diagram commutes in $\mathcal{D}_c$:

$$\begin{array}{ccc} FB & \xrightarrow{Ff} & FC \\ {\scriptstyle \alpha_B \triangleleft !_c}\downarrow & c & \downarrow{\scriptstyle \alpha_C \triangleleft !_c} \\ GB & \xrightarrow[Gf]{} & GC \end{array} \tag{4}$$

Notation III.5. We henceforth adopt the notation used in (4): when writing a diagram in a locally indexed category, we indicate the index by writing it in the centre of the shape.

We write $\mathbb{C}$-$\mathbf{LInd}$ for the 2-category of locally $\mathbb{C}$-indexed categories, locally $\mathbb{C}$-indexed functors, and transformations categories. As remarked above, this is exactly $\mathcal{P}(\mathbb{C})$-$\mathbf{Cat}$.

Definition III.6. A *locally $\mathbb{C}$-indexed adjunction* is an adjunction in the 2-category $\mathbb{C}$-$\mathbf{LInd}$. This is a pair of locally $\mathbb{C}$-indexed functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ with locally $\mathbb{C}$-indexed transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$ and $\varepsilon : \mathrm{id}_{\mathcal{D}} \Rightarrow FU$ satisfying the usual triangle equalities as composites in $\mathcal{D}_1$.

B. CBPV models

In order to model CBPV a locally indexed category needs to model the product and function types.

Definition III.7 (e.g. [36, §5])**.** Let $\mathbb{C}$ be a bicartesian category and $\mathcal{C}$ be a locally $\mathbb{C}$-indexed category.

1) $\mathcal{C}$ has *(countable) products* if for every countable family of objects $(B_i \mid i \in n)$ there exists an object $\prod_i B_i \in \mathcal{C}$ and arrows $\pi_i : \prod_i B_i \underset{1}{\to} B_i$ inducing an isomorphism $\mathcal{C}_c(A, \prod_i B_i) \cong \prod_i \mathcal{C}_c(A, B_i)$ natural in c and A.
2) $\mathcal{C}$ has *($\mathbb{C}$-indexed) powers* if for every $c \in \mathcal{C}$ and $B \in \mathcal{C}$ there exists an object $c \Rightarrow B \in \mathcal{C}$ and an arrow eval : $(c \Rightarrow B) \overset{c}{\to} B$ inducing an isomorphism $\mathcal{C}_{b \times c}(A, B) \cong \mathcal{C}_b(A, c \Rightarrow B)$ natural in c and A.
3) The coproducts in $\mathbb{C}$ are *distributive in* $\mathcal{C}$ if for all $a, b_i \in \mathbb{C}$ and $A, B \in \mathcal{C}$ the following map is invertible:

$$\begin{aligned} \mathcal{C}_{a \times \sum_i b_i}(A, B) &\to \textstyle\prod_i \mathcal{C}_{a \times b_i}(A, B) \\ f &\mapsto \big(f \triangleleft (\mathrm{id}_a \times \mathrm{inj}_i)\big)_i \end{aligned}$$

A CBPV model is now defined by taking the appropriate universally-defined structure for each CBPV construct.

Definition III.8 (e.g. [36, §5])**.** A *CBPV model* consists of a bicartesian category $\mathbb{C}$ and a locally $\mathbb{C}$-indexed adjunction $F : \mathsf{self}\,\mathbb{C} \leftrightarrows \mathcal{C} : U$ such that $\mathcal{C}$ has products and powers, and the coproducts in $\mathbb{C}$ are distributive in $\mathcal{C}$.

For the sake of exposition, in this paper we will focus on relatively simple classes of CBPV models. We refer to [36, p. 85] and [35] for the details of these and many other models.

Example III.9 ([36, §7])**.** As expected, the syntax forms a model. For any *signature* $\mathcal{S}$ of value base types, computation base types, and operations one may freely generate a *theory* and its classifying syntactic model $\mathbf{Syn}_{\mathcal{S}}$.

Example III.10 (Algebra models)**.** Let $\mathbb{C}$ be a biCCC and (T, t) a strong monad (see e.g. [14], [15]) on $\mathbb{C}$. The category $\mathbb{C}^T$ of T-algebras becomes a locally $\mathbb{C}$-indexed category $\mathcal{EM}(T)$ by taking maps $(A, a) \underset{c}{\to} (B, b)$ to be maps $c \times A \to B$ that are *right linear* [39]. The free–forgetful adjunction then becomes a CBPV model $F^T : \mathsf{self}\,\mathbb{C} \leftrightarrows \mathcal{EM}(T) : U^T$.

Example III.11 (Storage)**.** Let $(\mathbb{C}, \times, 1, \Rightarrow, 0, +)$ be a biCCC and $S \in \mathbb{C}$ be a fixed object of "states". The adjunction $(-) \times S \dashv S \Rightarrow (-)$ defines a CBPV model $\mathsf{self}\,\mathbb{C} \leftrightarrows \mathsf{self}\,\mathbb{C}$.

IV. CBPV FIBRATIONS

In this section we introduce CBPV fibrations. Our approach closely follows the pattern for monadic models of CBV in the style of Moggi [13], [12] pioneered by Katsumata [8], [9], [16] (see also [40], [41], [42], [10]). We begin by defining a 2-category of locally indexed categories (Section IV-A), and identify the *locally indexed fibrations* as fibrations internal to this 2-category (Section IV-B). In the locally indexed setting, these play the same role that traditional fibrations do in the non-indexed setting. We shall then define a CBPV fibration to be a locally indexed fibration that strictly preserves the structure of a CBPV model (Section IV-D). Along the way we shall also see that the general theory leads directly to a variety of examples, in the form of locally indexed versions of the codomain and subobject fibrations.

A. The 2-category $\mathbf{LInd}$

We begin by defining the 2-category of locally indexed categories. The objects are pairs $(\mathbb{C}, \mathcal{C})$ consisting of a cartesian category $\mathbb{C}$ and a locally $\mathbb{C}$-indexed category $\mathcal{C}$.

To define the 1-cells we need to allow the indexing categories to vary. To this end, observe that any cartesian functor $f : \mathbb{V} \to \mathbb{W}$ induces a cartesian functor $f^* : \mathcal{P}(\mathbb{W}) \to \mathcal{P}(\mathbb{V})$ by precomposition, and hence—by *change of base* (e.g. [43, §6.4])—a 2-functor $\mathbb{W}$-$\mathbf{LInd} \to \mathbb{V}$-$\mathbf{LInd}$ which we also denote by f^*. Explicitly, if $\mathcal{C} \in \mathbb{W}$-$\mathbf{LInd}$ then $f^*\mathcal{C}$ has the same objects and hom-presheaves defined by $(f^*\mathcal{C})_V := \mathcal{C}_{fV}$. Composition and identities are as in $\mathcal{C}$, and reindexing along ρ in $f^*\mathcal{C}$ is given by reindexing along $f(\rho)$ in $\mathcal{C}$.

We may now define a *locally indexed functor* $(f, F) : (\mathbb{C}, \mathcal{C}) \to (\mathbb{D}, \mathcal{D})$ to be a cartesian functor f together with a locally $\mathbb{C}$-indexed functor $F : \mathcal{C} \to f^*\mathcal{D}$. This smoothly handles reindexing: a map $k \in \mathcal{C}_c(C, C')$ is sent to a map $Fk \in (f^*\mathcal{D})_c(FC, FC') = \mathcal{D}_{fc}(FC, FC')$.

The 2-cells are defined similarly. Indeed, both change-of-base and the passage from functors f between categories to functors f^* between presheaf categories are 2-functorial [44], [45]. Thus, every natural transformation $\gamma : f \Rightarrow g$ defines a strict natural transformation $\gamma^* : g^* \Rightarrow f^* : \mathbb{W}$-$\mathbf{LInd} \to \mathbb{V}$-$\mathbf{LInd}$. The component $(\gamma^*)_{\mathcal{C}}$ at $\mathcal{C} \in \mathbb{W}$-$\mathbf{LInd}$ is the identity-on-objects $\mathbb{V}$-$\mathbf{LInd}$-functor which reindexes along γ:

$$(\gamma^*)_{\mathcal{C}}(V) := (g^*\mathcal{C})_V = \mathcal{C}_{gV} \xrightarrow{(-) \triangleleft \gamma_V} = \mathcal{C}_{fV} = (f^*\mathcal{C})_V$$

We define a *locally indexed 2-cell* $(f, F) \Rightarrow (g, G) : (\mathbb{C}, \mathcal{C}) \to (\mathbb{D}, \mathcal{D})$ to be a natural transformation $\gamma : f \Rightarrow g$ together with a locally $\mathbb{C}$-indexed transformation $\overline{\gamma} : F \Rightarrow (\alpha^*)_{\mathcal{D}} \circ G$. Concretely, $\overline{\gamma}$ is a family $\big(\overline{\gamma}_C : FC \underset{1}{\to} GC\big)_{C \in \mathcal{C}}$ such that for any $k : C \underset{c}{\to} C'$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{F(k)} & FC' \\ {\scriptstyle \overline{\gamma}_C \triangleleft !_c}\downarrow & f(c) & \downarrow{\scriptstyle \overline{\gamma}_{C'} \triangleleft !_c} \\ GC & \xrightarrow[G(k) \triangleleft \gamma_c]{} & GC' \end{array} \qquad (5)$$

Definition IV.1. We write $\mathbf{LInd}$ for the 2-category of locally indexed categories, locally indexed functors, and locally indexed transformations.

Remark IV.2. Abstractly, $\mathbf{LInd}$ is the 2-Grothendieck construction [46], [47] of the 2-functor $K : \mathbf{DistCat}^{\mathrm{coop}} \to 2\text{-}\mathbf{Cat}$ defined on objects by $K(\mathbb{V}) := \mathbb{V}$-$\mathbf{LInd}$ and on 1-cells and 2-cells by change of base as in Section IV-A.

B. Locally indexed fibrations

We are now in a position to identify *locally indexed fibrations* as fibrations internal to $\mathbf{LInd}$. We do this using [30, Theorem 2.7]. We therefore need to construct comma objects.

Construction IV.3. Let $(\mathbb{A}, \mathcal{A}) \xrightarrow{(f,F)} (\mathbb{C}, \mathcal{C}) \xleftarrow{(g,G)} (\mathbb{B}, \mathcal{B})$ be a cospan in $\mathbf{LInd}$. The comma object is defined as follows. The indexing category is the comma object $(f \downarrow g)$ in $\mathbf{CartCat}$ (Example II.7). Objects in $(F \downarrow G)$ are triples $\big(A \in \mathcal{A}, B \in \mathcal{B}, k : FA \underset{1}{\to} GB\big)$. A map $(A, B, k) \underset{j}{\to} (A', B', k')$ over $j : fa \to gb$ consists of maps $u : A \underset{a}{\to} A'$ and $v : B \underset{b}{\to} B'$ such that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{F(u)} & FA' \\ {\scriptstyle k \triangleleft !_{fa}}\downarrow & f(a) & \downarrow{\scriptstyle k' \triangleleft !_{fa}} \\ GB & \xrightarrow[G(v) \triangleleft j]{} & GB' \end{array}$$

Composition, identities, and reindexing are componentwise.

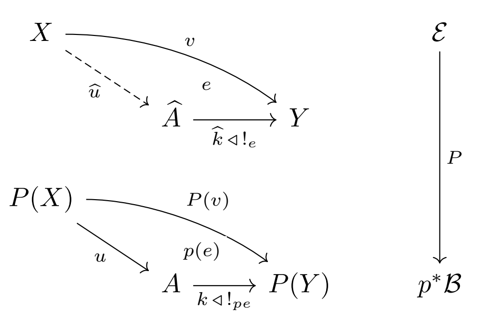
As an immediate corollary we also obtain pullbacks so long as they exist in $\mathbf{CartCat}$ (Example II.9).

Construction IV.4. Let $(\mathbb{A}, \mathcal{A}) \xrightarrow{(f,F)} (\mathbb{C}, \mathcal{C}) \xleftarrow{(g,G)} (\mathbb{B}, \mathcal{B})$ be a cospan in $\mathbf{LInd}$ such that the pullback $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$ exists in $\mathbf{CartCat}$. Then the pullback of this cospan exists in $\mathbf{LInd}$, and is defined by restricting $F \downarrow G$ to the objects (A, B) such that $F(A) = G(B)$.

Similarly, one may construct the product $(\mathbb{C}, \mathcal{C}) \times (\mathbb{D}, \mathcal{D})$ in $\mathbf{LInd}$ by first taking the product $\mathbb{C} \times \mathbb{D}$ in $\mathbf{CartCat}$ and then defining $\mathcal{C} \times \mathcal{D}$ to have objects pairs $(C \in \mathcal{C}, D \in \mathcal{D})$ and hom-presheaves $(\mathcal{C} \times \mathcal{D})_{(c,d)} := \mathcal{C}_c \times \mathcal{D}_d$. In fact, each of these constructions may be seen as following directly from Remark IV.2 and the fact that the Grothendieck 2-category for a 2-functor $K : \mathcal{C}^{\mathrm{coop}} \to 2\text{-}\mathbf{Cat}$ has those limits which exist in $\mathcal{C}$ and every 2-category $K(A)$, and are preserved by every $F(f)$ (cf. the construction of limits in the total category of a fibration [29, §4]).

Now, working through the condition in [30, Theorem 2.7] yields the following. To simplify notation we elide the iso $p(1) \cong 1$ given by the fact p is cartesian.

Proposition IV.5. *A locally indexed functor* $(p, P) : (\mathbb{E}, \mathcal{E}) \to (\mathbb{B}, \mathcal{B})$ *is an internal fibration if and only if* p *is a fibration and* P *satisfies the following lifting property. For every* $k : A \underset{1}{\to} PY$ *in* $\mathcal{B}$ *there exists an object* $\widehat{A} \in \mathcal{B}$ *and an arrow* $\widehat{k} : \widehat{A} \underset{1}{\to} Y$ *in* $\mathcal{E}$ *such that, for any triangle downstairs as shown in the next diagram, there exists a unique lift* $\widehat{u}$ *making the triangle upstairs commute:*



Definition IV.6. A *locally indexed fibration / opfibration / bifibration* is a fibration / opfibration / bifibration in $\mathbf{LInd}$.

Remark IV.7. This definition is not immediately obvious. For example, when writing a definition of locally indexed fibration by hand one might be tempted to set the universal property so that maps over any index have a cartesian lift. Nonetheless, our definition fits into the schema of locally indexed universal properties: compare it to the way the universal arrow defining a product—namely, the projections—lies over 1.

Because our definition arises from the mathematical theory we immediately know it is robust, and satisfies useful properties such as closure under pullback. Moreover, we can use it to define locally indexed versions of the codomain and subobject fibrations. Thus, we also get locally indexed versions of the core building blocks for constructing new models. The construction of the codomain opfibration follows directly from Construction IV.3 and Example II.8.

Construction IV.8. The *locally indexed arrow category* of a locally indexed category $(\mathbb{C}, \mathcal{C})$ is the $\mathbb{C}^{\to}$-indexed category with objects arrows $A \underset{1}{\to} B$ in $\mathcal{C}_1$. There is a canonical locally indexed *codomain* functor $(\mathrm{cod}, \mathrm{cod}) : (\mathbb{C}^{\to}, \mathcal{C}^{\to}) \to (\mathbb{C}, \mathcal{C})$.

This leads naturally to the subobject fibration.

Definition IV.9. We write $\mathrm{Sub}(\mathcal{C})$ for the $\mathrm{Sub}(\mathbb{C})$-indexed category obtained by restricting the objects of $\mathcal{C}^{\to}$ to arrows $A \underset{1}{\to} B$ that are monic in $\mathcal{C}_1$. Since $\mathrm{Sub}(\mathbb{C})$ is closed under products, reindexing is as in $\mathcal{C}^{\to}$.

In $\mathbf{Cat}$, the codomain functor is a fibration if and only if the base category $\mathbb{C}$ has pullbacks; then $\mathrm{cod} : \mathrm{Sub}\,\mathbb{C} \to \mathbb{C}$ is a also fibration. A corresponding fact is true here.

Definition IV.10. A locally indexed category $(\mathbb{C}, \mathcal{C})$ *has locally indexed pullbacks* if $\mathcal{C}_1$ has pullbacks, which are preserved by every $(-) \triangleleft !_c$.

Lemma IV.11. *Let* $(\mathbb{C}, \mathcal{C})$ *be a locally indexed category. The locally indexed codomain functor is a locally indexed bifibration if and only if* $\mathcal{C}$ *has with locally indexed pullbacks. In this situation, the fibration structure restricts to make* $(\mathrm{Sub}\,\mathbb{C}, \mathrm{Sub}\,\mathcal{C}) \to (\mathbb{C}, \mathcal{C})$ *a fibration as well.*

Example IV.12. Suppose $\mathbb{C}$ is finitely complete. Then $\mathrm{self}\,\mathbb{C}$ has locally indexed pullbacks. Moreover, for any strong monad (T, t) on $\mathbb{C}$ the induced locally $\mathbb{C}$-indexed category $\mathcal{EM}(T)$ of T-algebras also has locally indexed pullbacks.

C. The 2-category of CBPV models

Now we have locally indexed fibrations in hand, all that remains is to define a 2-category of CBPV models. We will then be able to say a CBPV fibration is a "map of CBPV models that preserve all the structure". In this section we isolate a 2-category of CBPV models and lax maps as a sub-2-category of $\mathrm{Adj}(\mathbf{LInd})_{\mathrm{lx}}$. In Section IV-D we will combine this with Definition IV.6 to define CBPV fibrations.

Every CBPV model is an object in $\mathrm{Adj}(\mathbf{LInd})$ of a special kind, because we require the domain of the left adjoint to be of the form $\mathsf{self}\,\mathbb{C}$. In other words, this part of the data is wholly determined by a bicartesian category $\mathbb{C}$. Correspondingly, we will isolate the maps of CBPV models as the maps in $\mathrm{Adj}(\mathbf{LInd})_{\mathrm{lx}}$ for which the action on this component is determined by bicartesian structure. For this we use the following lemma.

Lemma IV.13. *The* self *construction (Example III.2) extends to a 2-functor* $\mathbf{DistCat} \to \mathbf{LInd}$*. Moreover, this preserves products, comma objects, pullbacks whenever they exist, and fibrations which strictly preserve products.*

The 1-cells in our 2-category $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ of CBPV models are the ones we wish to pull back along in our lifting theorem. Thus, we only ask for preservation of products: this matches the situation for CBV models, where one can construct new models by pulling back along cartesian—not just cartesian closed—functors (cf. [16, Proposition 6]).

Definition IV.14. A locally $\mathbb{C}$-indexed functor $F : \mathcal{C} \to \mathcal{D}$ *preserves* I*-ary products* if the canonical map $\langle F\pi_i \rangle_{i \in I} : F(\prod_i C_i) \underset{1}{\to} \prod_i F(C_i)$ is an isomorphism in $\mathcal{D}_1$. It preserves products *strictly* if all the structure is preserved on the nose:

$$F(\textstyle\prod_i C_i) = \prod_i FC_i \quad F\pi_i = \pi_i \quad F\langle f_i \rangle = \langle F f_i \rangle$$

The definition of strict preservation of powers is likewise.

We now give the definition. The objects of $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ are CBPV models $(\mathbb{C}, \mathcal{C}, F, U)$. A 1-cell $(\mathbb{C}, \mathcal{C}, F^{\mathcal{C}}, U^{\mathcal{C}}) \to (\mathbb{D}, \mathcal{D}, F^{\mathcal{D}}, U^{\mathcal{D}})$ consists of

- A bicartesian functor $h : \mathbb{C} \to \mathbb{D}$,
- A locally $\mathbb{C}$-indexed functor $H : \mathcal{C} \to h^*\mathcal{D}$, and
- Locally $\mathbb{C}$-indexed transformations α and β,

such that H preserves products and $\big(\mathsf{self}\, f, H, (\mathrm{id}, \alpha), (\mathrm{id}, \beta)\big)$ is a 1-cell in $\mathrm{Adj}(\mathbf{LInd})$ as shown:

$$\begin{array}{ccccc} \mathsf{self}\,\mathbb{C} & \xrightarrow{(\mathrm{id}_{\mathbb{C}}, F^{\mathcal{C}})} & \mathcal{C} & \xrightarrow{(\mathrm{id}_{\mathbb{C}}, U^{\mathcal{C}})} & \mathsf{self}\,\mathbb{C} \\ {\scriptstyle \mathsf{self}\,h}\downarrow & (\mathrm{id}, \alpha)\Downarrow & \downarrow{\scriptstyle (h,H)} & \Downarrow(\mathrm{id}, \beta) & \downarrow{\scriptstyle \mathsf{self}\,h} \\ \mathsf{self}\,\mathbb{D} & \xrightarrow[(\mathrm{id}_{\mathbb{D}}, F^{\mathcal{D}})]{} & \mathcal{D} & \xrightarrow[(\mathrm{id}_{\mathbb{D}}, U^{\mathcal{D}})]{} & \mathsf{self}\,\mathbb{D} \end{array} \qquad (6)$$

Thus, for each $c \in \mathbb{C}$ and $C \in \mathcal{C}$ we have arrows

$$\alpha_c : (HF^{\mathcal{C}})c \underset{1}{\to} (F^{\mathcal{D}}h)c \qquad \beta_C : (hU^{\mathcal{C}})C \underset{1}{\to} (U^{\mathcal{D}}H)C$$

natural in the sense of (5) and satisfying the compatibility axioms (3) as composites over 1.

Finally, a 2-cell $(h, H, \alpha, \beta) \Rightarrow (h', H', \alpha', \beta')$ in $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ consists of a natural transformation $\gamma : h \Rightarrow h'$ and a locally $\mathbb{B}$-indexed transformation $\overline{\gamma} : H \Rightarrow H'$ such that $(\mathsf{self}\,\gamma, \overline{\gamma})$ is a 2-cell in $\mathrm{Adj}(\mathbf{LInd})_{\mathrm{lx}}$.

D. Defining CBPV fibrations

We can finally define CBPV fibrations as strictly structure-preserving locally indexed fibrations. Note that we require p to be a *bi*fibration so that pullbacks along p exist in $\mathbf{DistCat}$.

Definition IV.15. A $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell (h, H, α, β) is *strict* if

1) h strictly preserves bicartesian structure,
2) H strictly preserves products and powers, and
3) (h, H) is a 1-cell in $\mathrm{Adj}(\mathbf{LInd})_{\mathrm{st}}$, i.e. α and β are both the identity.

A *CBPV fibration* (p, P) is a strict $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell such that (p, P) is a locally indexed fibration and p is a bifibration.

Example IV.16 (Recall Example III.9)**.** Levy's proof of [36, Proposition 7.3] essentially shows that for any CBPV model $(\mathbb{C}, \mathcal{C}, F, U)$ with an interpretation of the base types and operations in a signature $\mathcal{S}$ there exists a strict $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell $\mathbf{Syn}_{\mathcal{S}} \to (\mathbb{C}, \mathcal{C}, F, U)$ extending the interpretation of $\mathcal{S}$. Moreover, this is unique up to isomorphism.

By Lemma IV.13, a CBPV fibration is a transformation—i.e. a 1-cell in $[\mathbf{Adj}, \mathbf{LInd}]_{\mathrm{st}}$—in which each component is a locally indexed fibration.

Turning now to examples, one simple class of CBPV fibrations comes via monad liftings. A particular instance of the following result has been studied by Kammar [50, §9.2], who constructs CBPV fibrations over $\mathbf{Set}$ using the free lifting.

Lemma IV.17. *Let* $(\widehat{\mathbb{C}}, \widehat{T})$ *and* $(\mathbb{C}, T)$ *be Moggi-style CBV models, and* $p : \widehat{\mathbb{C}} \to \mathbb{C}$ *a CBV fibration. Then* p *extends to a fibration* $\tilde{p} : \widehat{\mathbb{C}}^{\widehat{T}} \to \mathbb{C}^{T}$*, and this makes* $(p, \tilde{p}) : \mathcal{EM}(\widehat{T}) \to \mathcal{EM}(T)$ *a CBPV fibration.*

A further set of examples corresponds to the classical fact that, if $\mathbb{C}$ is a cartesian closed category with pullbacks, then the codomain fibration over $\mathbb{C}$ is an STLC fibration. The key technical result is the following.

Lemma IV.18. *Let* $\mathbb{C}$ *be a cartesian category with pullbacks and* $\mathcal{C}$ *be a locally* $\mathbb{C}$*-indexed category with locally indexed pullbacks, products, and powers. Then* $\mathcal{C}^{\to}$ *and* $\mathrm{Sub}\,\mathcal{C}$ *both have products and powers, and the codomain locally indexed functors strictly preserve this structure.*

Now, the $(-)^{\to}$ operation is 2-functorial so from a CBPV model $(\mathbb{C}, \mathcal{C}, F, U)$ we obtain a lifted locally $\mathbb{C}^{\to}$-indexed adjunction $(\mathbb{C}^{\to}, \mathcal{C}^{\to}, F^{\to}, U^{\to})$. Combining the preceding lemma with the observation that $(\mathsf{self}\,\mathbb{C})^{\to} \cong \mathsf{self}\,(\mathbb{C}^{\to})$ in $\mathbb{C}^{\to}$-$\mathbf{LInd}$, we obtain the following.

Proposition IV.19. *Let* $(\mathbb{C}, \mathcal{C}, F, U)$ *be a CBPV model such that* $\mathbb{C}$ *has pullbacks and* $\mathcal{C}$ *has locally indexed pullbacks. Then the codomain functor* $(\mathrm{cod}, \mathrm{cod}) : (\mathbb{C}^{\to}, \mathcal{C}^{\to}) \to (\mathbb{C}, \mathcal{C})$ *is a CBPV fibration.*

The only obstacle to applying a similar argument to the subobject fibration is that the left adjoint F may not preserve monics, and therefore may not restrict to a locally indexed functor $\mathsf{self}\,(\mathrm{Sub}\,\mathbb{C}) \to \mathrm{Sub}\,\mathcal{C}$ (right adjoints always preserve monics). This can be rectified by taking an appropriate factorisation system, in the style of [41], [51], [40], [42]. For reasons of space, however, we content ourselves to the case when F preserves monics. This turns out to be remarkably common: for example, it applies to the storage model of Example III.11), the erratic choice and continuation models of [36, §5.7], and any algebra model over $\mathbf{Set}$ (see [52, p. 89-90]).

Corollary IV.20. *In the situation of Proposition IV.19, if moreover F_1 preserves monics then the codomain functor* $(\mathrm{cod}, \mathrm{cod}) : (\mathrm{Sub}\,\mathbb{C}, \mathrm{Sub}\,\mathcal{C}) \to (\mathbb{C}, \mathcal{C})$ *is a CBPV fibration.*

Example IV.21. Consider the model of Example III.11 in the case where $\mathbb{C}$ has pullbacks. Since both the left and right adjoints preserve monics, this lifts to a Storage model $(-) * \overline{S} \dashv \overline{S} \supset (-)$ on $\mathrm{Sub}\,\mathbb{C}$ for any subobject $\overline{S} \rightarrowtail S$. Then the subobject locally indexed fibration is a CBPV fibration; Corollary IV.20 is the case where $\overline{S} := (S \xrightarrow{\mathrm{id}} S)$.

V. A LIFTING THEOREM FOR CBPV MODELS

We now have all the technology to present our central technical result: a lifting theorem for CBPV fibrations, paralleling those for STLC and CBV. The total space is defined as follows.

Definition V.1. A *CBPV lifting* is a CBPV fibration $(p, P) : (\widehat{\mathbb{C}}, \widehat{\mathcal{C}}, \widehat{F}, \widehat{U}) \to (\mathbb{C}, \mathcal{C}, F, U)$. A *map of liftings* $(p, P) \to (p', P')$ consists of $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$-morphisms $(\widehat{m}, \widehat{N}, \widehat{\alpha}, \widehat{\beta})$ and (m, N, α, β) as shown below, such that the following diagram commutes in $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$:

$$\begin{array}{ccc} (\widehat{\mathbb{C}},\widehat{\mathcal{C}},\widehat{F},\widehat{U}) & \xrightarrow{(\widehat{m},\widehat{N},\widehat{\alpha},\widehat{\beta})} & (\widehat{\mathbb{C}'},\widehat{\mathcal{C}'},\widehat{F'},\widehat{U'}) \\ {\scriptstyle (p,P)}\downarrow & & \downarrow{\scriptstyle (p',P')} \\ (\mathbb{C},\mathcal{C},F,U) & \xrightarrow[(m,N,\alpha,\beta)]{} & (\mathbb{C}',\mathcal{C}',F',U') \end{array} \qquad (7)$$

We write $\mathbf{CBPV}_{\mathrm{lift}}$ for the category of CBPV liftings and their maps, and $\mathrm{cod} : \mathbf{CBPV}_{\mathrm{lift}} \to \mathbf{CBPV}^{\times}_{\mathrm{lx}}$ for the codomain functor.

Theorem V.2. $\mathrm{cod} : \mathbf{CBPV}_{\mathrm{lift}} \to \mathbf{CBPV}^{\times}_{\mathrm{lx}}$ *is a fibration.*

We now recover the situation discussed in the introduction, as we now explain. Let $(p, P) : (\widehat{\mathbb{D}}, \widehat{\mathcal{D}}, F^{\widehat{\mathcal{D}}}, U^{\widehat{\mathcal{D}}}) \to (\mathbb{D}, \mathcal{D}, F^{\mathcal{D}}, U^{\mathcal{D}})$ be a CBPV fibration. Also suppose that we have a $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell (h, H, α, β) as in (8). Then there exists a universal choice of CBPV model and a CBPV fibration (q, Q) as shown:

$$\begin{array}{ccc} (\widehat{\mathbb{C}},\widehat{\mathcal{C}},F^{\widehat{\mathcal{C}}},U^{\widehat{\mathcal{C}}}) & \overset{(\widehat{h},\widehat{H},\widehat{\alpha},\widehat{\beta})}{\dashrightarrow} & (\widehat{\mathbb{D}},\widehat{\mathcal{D}},F^{\widehat{\mathcal{D}}},U^{\widehat{\mathcal{D}}}) \\ {\scriptstyle (q,Q)}\downarrow & & \downarrow{\scriptstyle (p,P)} \\ (\mathbb{C},\mathcal{C},F^{\mathcal{C}},U^{\mathcal{C}}) & \xrightarrow[(h,H,\alpha,\beta)]{} & (\mathbb{D},\mathcal{D},F^{\mathcal{D}},U^{\mathcal{D}}) \end{array} \qquad (8)$$

This is an instance of a general fact about lax transformations: see Section VIII. We sketch the concrete construction. First, q and Q are defined as pullbacks in $\mathbf{DistCat}$ and $\mathbf{LInd}$ respectively; these exist because p is strict and a bifibration (Example II.9 and Construction IV.8).

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\widehat{h}} & \widehat{\mathbb{D}} \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow[h]{} & \mathbb{D} \end{array} \qquad \begin{array}{ccc} (\widehat{\mathbb{C}},\widehat{\mathcal{C}}) & \xrightarrow{(\widehat{h},\widehat{H})} & (\widehat{\mathbb{D}},\widehat{\mathcal{D}}) \\ {\scriptstyle (q,Q)}\downarrow & \lrcorner & \downarrow{\scriptstyle (p,P)} \\ (\mathbb{C},\mathcal{C}) & \xrightarrow[(h,H)]{} & (\mathbb{D},\mathcal{D}) \end{array} \qquad (9)$$

An argument similar to that for cartesian closed structure (e.g. [16, Proposition 6]) shows $(\widehat{\mathbb{C}}, \widehat{\mathcal{C}})$ has products and powers. We define the adjunction $(\mathsf{self}\,\widehat{\mathbb{C}} \leftrightarrows \widehat{\mathcal{C}})$ and the 2-cells $\widehat{\alpha}$ and $\widehat{\beta}$ in (8) using the universal property of the fibrations. Observe first that the following diagram commutes because self is a 2-functor and (p, P) is a strict adjunction morphism:

$$\begin{array}{ccccc} \mathsf{self}\,\widehat{\mathbb{C}} & \xrightarrow{\mathsf{self}\,\widehat{h}} & \mathsf{self}\,\widehat{\mathbb{D}} & \xrightarrow{F^{\widehat{\mathcal{D}}}} & (\widehat{\mathbb{D}},\widehat{\mathcal{D}}) \\ {\scriptstyle \mathsf{self}\,q}\downarrow & & \downarrow{\scriptstyle \mathsf{self}\,p} & & \downarrow{\scriptstyle (p,P)} \\ \mathsf{self}\,\mathbb{C} & \xrightarrow[\mathsf{self}\,h]{} & \mathsf{self}\,\mathbb{D} & \xrightarrow[F^{\mathcal{D}}]{} & (\mathbb{D},\mathcal{D}) \end{array}$$

For any $(c, \widehat{d}) \in \mathsf{self}\,\widehat{\mathbb{C}}$ we may therefore apply the universal property of the fibration (p, P) to $\alpha_{q(c,\widehat{d})} = \alpha_c$: For this, fix any object $(C, \widehat{D}) \in \widehat{\mathcal{C}}$ (recall Construction IV.8) and apply the universal property of the fibration to the arrow $\alpha_{q(c,\widehat{d})} = \alpha_c$:

$$\begin{array}{ccccc} \widehat{\alpha_c}(F^{\widehat{\mathcal{D}}}\widehat{h}c) & \overset{\widehat{\alpha}_c}{\underset{1}{\dashrightarrow}} & (F^{\widehat{\mathcal{D}}}\widehat{h})c & \qquad & \widehat{\mathcal{D}} \\ & & & & \downarrow{\scriptstyle (p,P)} \\ (HF^{\mathcal{C}})c & \xrightarrow[1]{\alpha_c} & (F^{\mathcal{D}}h)c = (PF^{\widehat{\mathcal{D}}}\widehat{h})c & & \mathcal{D} \end{array}$$

This definition extends to a locally indexed functor $K : \mathsf{self}\,\widehat{\mathbb{C}} \to (\widehat{\mathbb{D}}, \widehat{\mathcal{D}})$, so we may use the universal property of the pullback in (9) to define $F^{\widehat{\mathcal{C}}}(c, \widehat{d})$ as the unique locally indexed functor filling the next diagram:

$$\begin{array}{cccc} \mathsf{self}\,\widehat{\mathbb{C}} & \overset{F^{\widehat{\mathcal{C}}}}{\searrow} & \overset{K}{\longrightarrow} & \\ {\scriptstyle \mathsf{self}\,q}\downarrow & & & \\ \mathsf{self}\,\mathbb{C} & & (\widehat{\mathbb{C}},\widehat{\mathcal{C}}) \xrightarrow{(\widehat{h},\widehat{H})} & (\widehat{\mathbb{D}},\widehat{\mathcal{D}}) \\ & \overset{F^{\mathcal{C}}}{\searrow} & {\scriptstyle (q,Q)}\downarrow \quad \lrcorner & \downarrow{\scriptstyle (p,P)} \\ & & (\mathbb{C},\mathcal{C}) \xrightarrow[(h,H)]{} & (\mathbb{D},\mathcal{D}) \end{array}$$

The right adjoint $U^{\widehat{\mathcal{C}}}$ and 2-cell $\widehat{\beta}$ are constructed similarly.

Lifting via opfibrations. As a consequence of our general theory (see Section VIII), Theorem V.2 has a dual, as follows. Define a *CBPV oplifting* to be a strict $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell (p, P) such that p is a bifibration and P is an opfibration, and a map of opliftings to be a pair of $\mathbf{CBPV}^{\times}_{\mathrm{oplx}}$ 1-cells such that the diagram (7) commutes in $\mathbf{CBPV}^{\times}_{\mathrm{oplx}}$. We write $\mathbf{CBPV}_{\mathrm{oplift}}$ for the category of opliftings and their maps.

Corollary V.3. *The codomain functor* $\mathrm{cod} : \mathbf{CBPV}_{\mathrm{oplift}} \to \mathbf{CBPV}^{\times}_{\mathrm{oplx}}$ *is a fibration.*

Concretely the construction is similar to that outlined above, except $\widehat{\alpha}$ and $\widehat{\beta}$ are defined using *op*fibration structure.

Remark V.4. This theorem is useful in practical situations because in general a monad morphism $\sigma : S \to T$—that is, a natural transformation compatible with the units and multiplications (e.g. [49])—induces an *oplax* adjunction morphism from the Eilenberg–Moore adjunction of T to the Eilenberg–Moore adjunction of S [23]. In such situations

Corollary V.3 applies even though Theorem V.2 does not. For a concrete example, see Example VI.1.

A. *Examples*

In this section we sketch some simple applications of our theorem. We leave a detailed exploration of the models for elsewhere: the aim is simply to show how our theorem yields a framework for building CBPV models, just as previous results do this for STLC and CBV models (cf. e.g. [53], [5], [42]).

Example V.5. We start with the Storage model as in Example III.11. There is a lax adjunction morphism from the Storage model $(-) \times S \vdash S \Rightarrow (-)$ on $\mathbb{C}$ to the Storage model $(-) \times \mathbb{C}(1, S) \vdash \mathbb{C}(1, S) \Rightarrow (-)$ on $\mathbf{Set}$ as follows. The functors h and H are both given by $\mathbb{C}(1, -)$. The 2-cell α is the isomorphism $\mathbb{C}(1, - \times S) \cong \mathbb{C}(1, -) \times \mathbb{C}(1, S)$ while $\beta_A : \mathbb{C}(1, S \Rightarrow A) \to \big(\mathbb{C}(1, S) \Rightarrow \mathbb{C}(1, A)\big)$ sends t to $\lambda u \in \mathbb{C}(1, S) \,.\, \mathrm{eval} \circ \langle t, \mu\rangle$. The model in $\mathbf{Set}$ is easily lifted to $\mathbf{Pred}$: we take any subset $\overline{S} \subseteq \mathbb{C}(1, S)$ and consider the corresponding Storage model on $\mathbf{Pred}$ (Example IV.21). Applying our construction, we get a CBPV model indexed by the category $\widehat{\mathbb{C}}$ with objects pairs $\big(C \in \mathbb{C}, R \subseteq \mathbb{C}(1, X)\big)$. Since self commutes with pullbacks (Lemma IV.13), the locally indexed category must also be $\mathsf{self}\,\widehat{\mathbb{C}}$. The lifted left and right adjoints $\widehat{F}$ and $\widehat{U}$ are defined by $\widehat{F}(C, R) = (C \times S, R \times \overline{S})$ and $\widehat{F}(C, R) = (S \Rightarrow C, \overline{S} \supset R)$

Next we use the universal property of the syntactic model (Example IV.16) to recover a definition of CBPV logical relations in the syntactic style. More precisely, from purely semantic reasoning we recover a version of the logical relations used by McDermott [54, p. 114].

Example V.6. Let $\mathcal{S}$ be a signature of base types and basic operations, and let T be the free monad on $\mathbf{Set}$ which supports these operations, so that that the algebra model $F^T : \mathsf{self}\,\mathbf{Set} \leftrightarrows \mathbf{Set}^T : U^T$ is a sound model of CBPV with base types and operations in $\mathcal{S}$. Such a monad always exists: one sends a set X to the set of terms generated using the basic operations with variables in X (cf. [36, Remark 7.2]).

Now define an interpretation of base types and operations in $(\mathbf{Set}, \mathbf{Set}^T, F^T, U^T)$ by setting the interpretation of a value type A to be the set of closed value terms of type A, and the interpretation of a computation type $\overline{A}$ to be the set of closed computations of type $\overline{A}$. By the free property of $\mathbf{Syn}_{\mathcal{S}}$, this extends to a strict map $\mathbf{Syn}_{\mathcal{S}} \to \big(\mathbf{Set}, \mathcal{EM}(T), F^T, U^T\big)$.

Finally, let $\widehat{T}$ be a lifting of T to $\mathbf{Pred}$; for definiteness, we choose the *free lifting* [50], [42]. Now apply Lemma IV.17 and Theorem V.2 to obtain a model $(\widehat{\mathbb{C}}, \widehat{\mathcal{C}}, \widehat{F}, \widehat{U})$ as shown:

$$\begin{array}{ccc} (\widehat{\mathbb{C}}, \widehat{\mathcal{C}}, \widehat{F}, \widehat{U}) & \dashrightarrow & \big(\mathsf{self}\,\mathbf{Pred} \leftrightarrows \mathcal{EM}(\widehat{T})\big) \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{cod}, \mathrm{cod})} \\ \mathbf{Syn}(\mathcal{S}) & \xrightarrow[\exists!]{} & \big(\mathsf{self}\,\mathbf{Set} \leftrightarrows \mathcal{EM}(T)\big) \end{array}$$

Objects in $\widehat{\mathbb{C}}$ consist of a value type A and a set V_A of closed value terms of type A. Objects in $\widehat{\mathcal{C}}$ consist of a computation type $\overline{A}$ and a set of $C_{\overline{A}}$ of closed terms of type $\overline{A}$ which is further equipped with $\widehat{T}$-algebra structure. The action of the adjoints $\widehat{F}$ and $\widehat{U}$ in the lifted model are as follows:

$$\widehat{F}(A, V_A) = \big(FA, F^{\widehat{T}} V_A\big) \quad \widehat{U}(\overline{B}, C_{\overline{A}}) = \big(U\overline{B}, U^{\widehat{T}} C_{\overline{A}}\big)$$

Since $\widehat{T}$ is the free lifting, $\widehat{F}(A, V_A)$ consists of the type FA and the smallest relation containing V_A that is closed under $\mathtt{return}$ and the operations in $\mathcal{S}$. On the other hand, $\widehat{U}(\overline{B}, C_{\overline{A}})$ consists of the type $U\overline{B}$ and the set $C_{\overline{A}}$ with its algebra structure forgotten; this reflects the fact that force is invisible in CBPV models (see [38, §15.1]). The action on products, sums, and exponentials is exactly as given by McDermott.

Our final example is a version of Katsumata's $\top\top$-lifting [8], adapted for CBPV models. Katsumata's construction relies on the fact that for any strong monad (T, t) and any T-algebra there is a canonical strong monad morphism into the corresponding continuation monad. Because monad morphisms induce adjunction morphisms contravariantly (Remark V.4), this approach is not immediately available for adjunction models. Our strategy, therefore, is to first pass from our starting CBPV model to its corresponding algebra model, and then ask for a lifting of that model via Lemma IV.17.

In the next example we focus on $\top\top$-lifting but the construction is parametric in this choice: the argument works verbatim for any other lifting (e.g. the free lifting [50], [42], codensity lifting [7] or the monadic lifting of [41], [40]).

Construction V.7 ($\top\top$-lifting for CBPV)**.** Let $(\mathbb{C}, \mathcal{C}, F, U)$ be a CBPV model in which $\mathbb{C}$ is also cartesian closed. Write T for the induced (strong) monad UF on $\mathbb{C}$. By [35, §11.6.2] there is a strict map into the Eilenberg–Moore model $\big(\mathbb{C}, \mathcal{EM}(T), F^T, U^T\big)$ for T. Now fix an STLC fibration $p : \mathbb{E} \to \mathbb{C}$—for example, the subobject fibration— and an object $R \in \mathbb{E}$ as a *lifting parameter*. Finally, let $\widehat{T}$ be the $\top\top$-lifting of T with this parameter. By Lemma IV.17, we obtain a CBPV fibration $\big(\mathbb{E}, \mathcal{EM}(\widehat{T}), F^{\widehat{T}}, U^{\widehat{T}}\big) \to \big(\mathbb{C}, \mathcal{EM}(T), F^T, U^T\big)$ and hence, by Theorem V.2, a lifted model as shown below:

$$\begin{array}{ccc} (\widehat{\mathbb{C}}, \widehat{\mathcal{C}}, \widehat{F}, \widehat{U}) & \dashrightarrow & \big(\mathbb{E}, \mathcal{EM}(\widehat{T}), F^{\widehat{T}}, U^{\widehat{T}}\big) \\ \downarrow & & \downarrow{\scriptstyle (p, \tilde{p})} \\ (\mathbb{C}, \mathcal{C}, F, U) & \longrightarrow & \big(\mathbb{C}, \mathcal{EM}(T), F^T, U^T\big) \end{array} \tag{10}$$

We call this the $\top\top$-*lifting* of the starting model.

Example V.8. We construct the $\top\top$-lifting of Levy's model of erratic choice [35, §5.5]. Thus, in our starting model the category of values is $\mathbf{Set}$ and the adjunction is the Kleisli resolution $J : \mathbf{Set} \leftrightarrows \mathbf{Rel} : K$ of the powerset monad $\mathcal{P}$. The induced monad on $\mathbf{Set}$ is also $\mathcal{P}$. To lift this to $\mathbf{Pred}$, we take as our lifting parameter the $\mathcal{P}$-algebra $\big(\{\{0, 1\}, \{1\}\}, \mathbb{T}\big)$ where $\mathbb{T} : \mathcal{P}(\{\{0, 1\}, \{1\}\}) \to \{\{0, 1\}, \{1\}\}$ maps $p \subseteq \{0, 1\}$ to 1 if $1 \in p$ and 0 otherwise. Applying $\top\top$-lifting, we obtain a strong monad $\widehat{\mathcal{P}}$ on $\mathbf{Pred}$. This acts as $\widehat{\mathcal{P}}(A, R) := (\mathcal{P}A, \widehat{\mathcal{P}}R)$ where $p \in \widehat{\mathcal{P}}R$ if and only if for all $f : X \to 2$ satisfying $\forall x \in R \,.\, f(x) = 1$ we have $\sum_{x \in X} f(x)\, p(x) = 1$. A direct calculation then shows that $p \in \widehat{\mathcal{P}}R$ if and only if every $x \in p$ is in R. Applying our $\top\top$-lifting construction (10), the resulting model is the Kleisli adjunction $\mathbf{Pred} \leftrightarrows \mathbf{Pred}_{\widehat{\mathcal{P}}}$ for $\widehat{\mathcal{P}}$.

VI. Effect Simulation

In this section we use our CBPV-fibration theorem to give a semantic account to the effect simulation problem for languages based on CBPV. The effect simulation problem is about relating different denotational interpretations for the same computational effects. For example, it is possible to use both the finite powerset and list monads to give semantics to non-deterministic computation. An effect simulation property relating both semantics would say, for instance, that the set of possible elements denoted by the list and the powerset semantics are the same.

This problem has been thoroughly studied in the context of Moggi's metalanguage. In this section we show how our theory gives a semantic account of effect simulation for CBPV, via an approach similar to Katsumata's for CBV [16]. The key idea is that effect simulation is about constructing a non-standard model over the product of the models we are trying to relate. Since $\mathbf{LInd}$ has products, which are preserved by self, and the 2-functor $(-) \times (=)$ preserves adjunctions, for any CBPV models $(\mathbb{C}_i, \mathcal{C}_i, F_i, U_i)$ we get a *product CBPV model* $\left(\prod_i \mathbb{C}_i, \prod_i \mathcal{C}_i, \prod_i F_i, \prod_i U_i\right)$.

Semantic effect simulation now arises from Theorem V.2 as follows. We start with two CBPV models, which for brevity we denote $\underline{\mathcal{C}}_i := (\mathbb{C}_i, \mathcal{C}_i, F_i, U_i)$ for $i = 1, 2$, a CBPV fibration (p, P), and a $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell as shown:

$$\begin{array}{ccc} & & (\mathbb{E}, \mathcal{E}, F^{\mathcal{E}}, U^{\mathcal{E}}) \\ & & \big\downarrow{\scriptstyle (p,P)} \\ \underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2 & \xrightarrow[(h,H,\alpha,\beta)\times \mathrm{id}]{} & \underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2 \end{array}$$

The effect simulation model is then constructed by applying Theorem V.2 or Corollary V.3 (cf. (8)).

Example VI.1 (Lists and powersets for non-determinism)**.** Consider the Eilenberg–Moore models for the finite powerset $\mathcal{P}_f$ and list $[-]$ monads over $\mathbf{Set}$. Their categories of algebras are, respectively, the category $\mathbf{SLat}$ of sup-semilattices and $\mathbf{Mon}$ of monoids. Since every adjunction in $\mathbf{Set}$ lifts to a $\mathbf{LInd}$-adjunction, in order to simplify the calculations, we will compute everything in terms of $\mathbf{Cat}$-adjunctions.

There is a canonical monad morphism $\gamma : [-] \to \mathcal{P}_{fin}$ which maps a list to its set of elements. This gives rise to an oplax adjunction morphism (recall Remark V.4) as shown:

$$\begin{array}{ccccc} \mathbf{Set} & \xrightarrow{\mathcal{P}_f} & \mathbf{SLat} & \xrightarrow{U} & \mathbf{Set} \\ {\scriptstyle \mathrm{id}}\big\downarrow & \overset{\gamma}{\nearrow} & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \mathbf{Set} & \xrightarrow[{[-]}]{} & \mathbf{Mon} & \xrightarrow[U]{} & \mathbf{Set} \end{array}$$

Next, we define our CBPV fibration. The lifted model over $(\mathbf{Set}\times\mathbf{Set}, \mathbf{Mon}\times\mathbf{Mon})$ is $(\mathbf{BPred}, \mathbf{BPredMon})$, defined as follows. $\mathbf{BPred}$ is the category of binary predicates: its objects are triples $(A \in \mathbf{Set}, B \in \mathbf{Set}, R \subseteq A \times B)$ and morphisms are pairs of functions that preserve the underlying binary relation. $\mathbf{BPredMon}$ is defined similarly, with the exception that the objects are monoids M and N, and the binary relation has to be a submonoid of $M \times N$. Applying Corollary V.3, we obtain a CBPV model $\mathbf{BPred} \leftrightarrows \mathbf{BPredSLatMon}$, where $\mathbf{BPredSLatMon}$ is defined as the following pullback:

$$\begin{array}{ccc} \mathbf{BPredSLatMon} & \longrightarrow & \mathbf{BPredMon} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{SLat} \times \mathbf{Mon} & \longrightarrow & \mathbf{Mon} \times \mathbf{Mon} \end{array}$$

The left adjoint acts on objects as $\widehat{F}(A, B, R) = \left(\mathcal{P}_f(A), [B], \widehat{R}\right)$, where for a finite set $p \subseteq A$ and list l over B, $(p, l) \in \widehat{R}$ if and only if for every $a \in p$ there is an element b in l such that $(a, b) \in R$, and, analogously, for every element b in l there's an element $a \in p$ such that $(a, b) \in R$. In this new model base types τ_b are interpreted as the diagonal relation $(b \times b, =)$ over an object b and the semantics of closed programs t of type $F\tau_b$ has the shape $(\gamma(l), l)$ for some list l.

VII. Relative full completeness

In this section we show how the our 2-categorical perspective leads relatively easily to a proof of *relative full completeness*, which establishes semantically that function types in CBPV are a conservative extension of the first-order fragment. Our proof follows the classic Lafont argument [56]: this argument is well-known, and has been applied in many differing situations (e.g. [57], [58], [59], [60]). Thus, our contribution here is not the proof strategy, but showing how to construct the ingredients to feed into the proof. Indeed, as several authors have independently noted [58], [59], the proof relies on having:

1) A suitable "presheaf" model and a "nerve" construction;
2) The existence of certain comma objects ("glueing").

In what follows we shall outline how each of these ingredients arises in the case of CBPV models. The rest of the argument follows the classical pattern, as in e.g. [57, §4.10] so, for reasons of space, we omit it.

As well as being of interest in its own right, we view the theory sketched here as a first step towards a semantic account of Kripke relations of varying arity for CBPV, and thereby a characterisation of definability (cf. [61], [2], [10]) and normalisation-by-evaluation in the style of [62].

We begin by constructing a version of presheaf models for CBPV by understanding the corresponding structure in $\mathbf{LInd}$.

A. Presheaf locally indexed categories

We first detail the abstract picture, then give the concrete definition. Our construction has two stages. First, as a 2-category of categories enriched in a presheaf category, each $\mathbb{C}$-$\mathbf{LInd}$ has a $\mathcal{P}(\mathbb{C})$-enriched presheaf construction. By general enriched-category theoretic considerations (e.g. [63,

§5.7]) this defines a pseudofunctor $\mathrm{P} : \mathbb{C}\text{-}\mathbf{LInd} \to \mathbb{C}\text{-}\mathbf{LIND}$ from the 2-category of small $\mathcal{P}(\mathbb{C})$-categories to the 2-category of large $\mathcal{P}(\mathbb{C})$-categories. On objects, $\mathrm{P}(\mathcal{C})$ is the $\mathcal{P}(\mathbb{C})$-enriched functor category $[\mathcal{C}, \mathcal{P}(\mathbb{C})]$. If $F : \mathcal{C} \to \mathcal{D}$ then $\mathrm{P}(F) := F_!$ is defined by left Kan extension; this also determines the action on transformations.

Applying P to $\mathcal{C} \in \mathbb{C}\text{-}\mathbf{LInd}$ yields a presheaf-like locally indexed category, but over the wrong base: it is still $\mathbb{C}$-indexed. However, the Yoneda functor is always cartesian, so we may apply change of base and define $\mathcal{P}$ as the composite

$$\mathbb{C}\text{-}\mathbf{LInd} \xrightarrow{\mathrm{P}} \mathbb{C}\text{-}\mathbf{LIND} \xrightarrow{y} \mathcal{P}\mathbb{C}\text{-}\mathbf{LIND} \tag{11}$$

Using standard enriched category-theoretic techniques, together with Levy's explicit identification of $\mathrm{P}(\mathcal{C})$ [38, p. 184], we arrive at the following characterisation of this composite.

Recall from e.g. [36, p. 84] that, for a locally $\mathbb{C}$-indexed category $\mathcal{C}$, the category $\mathsf{opGr}\,\mathcal{C}$ has objects $(c \in \mathbb{C}, C \in \mathcal{C})$ and morphisms $(d, C) \to (c, D)$ pairs of a map $\rho : c \to d$ in $\mathbb{C}$ and $f : C \underset{c}{\to} D$ in $\mathcal{C}$.

Definition VII.1. The *presheaf locally indexed category* $(\mathcal{P}\mathbb{C}, \mathcal{P}\mathcal{C})$ on $(\mathbb{C}, \mathcal{C}) \in \mathbf{LInd}$ is the $\mathcal{P}\mathbb{C}$-indexed category defined as follows. The objects are functors $\mathsf{opGr}\,\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, and maps $\tau : H \underset{P}{\to} H'$ are families of maps

$$\tau_{c,C} : P(c) \times H(c, C) \to H'(c, C)$$

natural in each argument. Composition, identities, and reindexing are as in $\mathsf{self}\,[\mathsf{opGr}\,\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

$(\mathcal{P}\mathbb{C}, \mathcal{P}\mathcal{C})$ is equivalently the locally $\mathcal{P}(\mathbb{C})$-indexed category obtained by reindexing $\mathsf{self}\,[\mathsf{opGr}\,\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ along the cartesian functor $\pi \circ (-) : [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathsf{opGr}\,\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ induced by the first projection $\pi : \mathsf{opGr}\,\mathcal{C} \to \mathbb{C}$. A short check shows that, if $\mathbb{C}$ and $\mathbb{D}$ are cartesian closed categories, and $f : \mathbb{C} \to \mathbb{D}$ preserves products, then $f^*(\mathsf{self}\,\mathbb{D}) \in \mathbb{C}\text{-}\mathbf{LInd}$ has products and $\mathbb{C}$-powers. Hence $\mathcal{P}\mathcal{C}$ has products and $\mathcal{P}(\mathbb{C})$-powers. Moreover, there exists an adjunction

$$[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] \leftrightarrows [\mathsf{opGr}\,(\mathsf{self}\,\mathbb{C})^{\mathrm{op}}, \mathbf{Set}]$$

in which the left adjoint acts by $P \mapsto P(- \times =)$ and the right adjoint acts by $H \mapsto H(-, 1)$. Using the explicit characterisation above, one sees this extends to a locally $\mathcal{P}(\mathbb{C})$-indexed adjunction $L : \mathsf{self}\,\mathcal{P}\mathbb{C} \leftrightarrows \mathcal{P}(\mathsf{self}\,\mathbb{C}) : R$.

B. Presheaf CBPV models

Because the composite $\mathcal{P}$ in (11) is pseudofunctorial, and pseudofunctors preserve adjunctions, for any CBPV model we get a new adjunction between the corresponding presheaf locally indexed categories. However, we must take some care: in $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ the morphisms consist of a *bicartesian* functor and a locally indexed functor but the Yoneda embedding does not generally preserve colimits. The fix is well-known (e.g. [64] or [65, §6]): if one restricts from all presheaves to finite product-preserving presheaves, the resulting presheaf category $\mathcal{P}^{\times}\mathbb{C}$ is still complete, cocomplete, and cartesian closed, and the Yoneda lemma still holds, but now the Yoneda embedding $y^{\times} : \mathbb{C} \to \mathcal{P}^{\times}\mathbb{C}$ also preserves finite coproducts. Modulo this adjustment, the theory above goes through verbatim.

Notation VII.2. For the rest of this section, we take $\mathcal{P}(\mathbb{C})$ to be the category of product-preserving presheaves, and y to be the restriction of the Yoneda functor to this subcategory.

Now consider any CBPV model $\underline{\mathcal{C}} := (\mathsf{self}\,\mathbb{C}, \mathcal{C}, F, U)$. Since pseudofunctors preserve adjunctions, we obtain a locally $\mathcal{P}(\mathbb{C})$-indexed adjunction $\mathcal{P}(\mathsf{self}\,\mathbb{C}) \leftrightarrows \mathcal{P}\mathcal{C}$ in which the adjoints $F_!$ and $U_!$ are computed using the left Kan extension in $\mathcal{P}(\mathbb{C})$-$\mathbf{Cat}$. We then define the *presheaf CBPV model* for $\underline{\mathcal{C}}$ to be the composite locally $\mathcal{P}(\mathbb{C})$-indexed adjunction

$$F_! \circ L : \mathsf{self}\,\mathcal{P}\mathbb{C} \leftrightarrows \mathcal{P}(\mathsf{self}\,\mathbb{C}) \leftrightarrows \mathcal{P}\mathcal{C} : R \circ U_!$$

We also obtain a Yoneda map into the presheaf model. Indeed, a short calculation shows that $(y)^*(\mathcal{P}(\mathcal{C}))$ is isomorphic to $[\mathcal{C}, \mathcal{P}(\mathbb{C})]$ in $\mathbb{C}\text{-}\mathbf{LIND}$. We therefore define a locally indexed map $(y, Y) : (\mathbb{C}, \mathcal{C}) \to (\mathcal{P}\mathbb{C}, \mathcal{P}\mathcal{C})$ by taking $Y : \mathcal{C} \to y^*(\mathcal{P}\mathcal{C})$ to be the $\mathcal{P}(\mathbb{C})$-enriched Yoneda embedding: $Y(C) := \mathcal{C}_{-}(=, C)$. This extends to a pseudonatural transformation $\mathrm{inc} \Rightarrow \mathrm{P}$ from the inclusion $\mathbb{C}\text{-}\mathbf{LInd} \hookrightarrow \mathbb{C}\text{-}\mathbf{LIND}$ to P (cf. [66, Lemma 3.7]) so there exists a pseudo adjunction map as in the right-hand square below; the left-hand square is a strict adjunction map.

$$\begin{array}{ccccc}
\mathsf{self}\,[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] & \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} & \mathcal{P}(\mathsf{self}\,\mathbb{C}) & \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} & (\mathcal{P}\mathbb{C}, \mathcal{P}\mathcal{C}) \\
{\scriptstyle \mathsf{self}\,y}\uparrow & & {\scriptstyle (y,Y)}\uparrow & \cong & \uparrow{\scriptstyle \mathsf{self}\,y} \\
\mathsf{self}\,\mathbb{C} & = & \mathsf{self}\,\mathbb{C} & \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} & (\mathbb{C}, \mathcal{C})
\end{array}$$

Altogether, we have shown the following proposition.

Definition VII.3. A locally indexed functor $(f, F) : (\mathbb{C}, \mathcal{C}) \to (\mathbb{D}, \mathcal{D})$ is *full / faithful / fully faithful* if both f and every functor $F_c : \mathcal{C}_c \to \mathcal{D}_{fc}$ are full / faithful / fully faithful. A $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell (f, F, α, β) is fully faithful if (f, F) is.

Proposition VII.4. *For any CBPV model $\underline{\mathcal{C}}$ there is a fully faithful $\mathbf{CBPV}^{\times}_{\mathrm{ps}}$ 1-cell $\underline{\mathcal{C}} \to \mathcal{P}\underline{\mathcal{C}}$ into the presheaf CBPV model. We denote this by $\underline{Y}$.*

Note that $\mathsf{self}\,f$ is fully faithful if f is. The final observation about presheaves we need is the following.

Proposition VII.5. *For any $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ morphism $\underline{F} : \underline{\mathcal{B}} \to \underline{\mathcal{C}}$ there exists a $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell $\underline{\langle F \rangle} : \underline{\mathcal{C}} \to \mathcal{P}\underline{\mathcal{B}}$ and a $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 2-cell $\Gamma : \underline{Y} \Rightarrow \underline{\langle F \rangle} \circ \underline{F}$.*

Indeed, for any locally indexed functor $(f, F) : (\mathbb{B}, \mathcal{B}) \to (\mathbb{C}, \mathcal{C})$ we obtain $\langle f \rangle : \mathbb{C} \to \mathcal{P}(\mathbb{B})$ and $\langle F \rangle : \mathcal{C} \to \langle f \rangle^*(\mathcal{P}\mathcal{B})$ by taking $\langle f \rangle c := \mathbb{C}(f-, c)$ and $\langle F \rangle(C) := \mathcal{C}_{f-}(=, C)$. Note that $\langle f \rangle$ preserves products because f does. The rest of the calculation essentially follows by unwinding the standard fact—which holds equally in the enriched setting—that $\langle f \rangle$ is the left Kan extension of y along f.

C. Completing the proof

Fix a signature $\mathcal{S}$ and let $\mathbf{Syn}_{\mathcal{S}}$ be the corresponding syntactic model (Example III.9). Also let $\mathbf{Syn}^{\mathrm{fo}}_{\mathcal{S}}$ be the first-order syntactic model, with function types omitted. Both these models are free (Example IV.16) so there is a canonical strict $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell $i : \mathbf{Syn}^{\mathrm{fo}}_{\mathcal{S}} \to \mathbf{Syn}_{\mathcal{S}}$. We prove the following.

Theorem VII.6 (Relative full completeness)**.** *For any signature $\mathcal{S}$, the canonical strict $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ 1-cell $i : \mathbf{Syn}^{\mathrm{fo}}_{\mathcal{S}} \to \mathbf{Syn}_{\mathcal{S}}$ is fully faithful.*

The remaining difficulty lies in seeing that for any $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ morphism (g, G, α, β) the following comma object exists in $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$, i.e. CBPV models admit *glueing*:

$$\begin{array}{ccc} \underline{\mathcal{G}} & \longrightarrow & \underline{\mathcal{C}} \\ \downarrow & \overset{\Longleftarrow}{\scriptstyle\lambda} & \| \\ \underline{\mathcal{B}} & \xrightarrow[(g,G,\alpha,\beta)]{} & \underline{\mathcal{C}} \end{array}$$

This follows from two facts. First, for any 2-category D, if $\mathcal{C}$ has comma objects then $[\mathrm{D}, \mathcal{C}]_{\mathrm{lx}}$ also has comma objects of the shape above, computed component-wise (cf. [55, Prop. 4.6]). Since $\mathbf{LInd}$ has all comma objects, so does $\mathrm{Adj}(\mathbf{LInd})_{\mathrm{lx}}$. Second, a small adaptation of the classical proof (e.g. [57]) shows this restricts to $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$: when $\underline{\mathcal{C}}$ is a CBPV model with locally indexed pullbacks, $\underline{\mathcal{G}}$ is also a CBPV model.

The rest of the argument is as in the classical case (see e.g. [57, §4.10] or [59, §3.2]), observing that composition in $\mathbf{LInd}$ reduces to (1) composition of the functors in $\mathbf{DistCat}$ on the first component, and (2) on the second component, composition in $\mathbf{Cat}$ at each index.

VIII. Lifting theorems for arbitrary shapes

In this final technical section we sketch the proof of Theorem V.2 and Corollary V.3 as special cases of a general result which applies generally to any "shape" of model, including CBV models. The key idea is that, since $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ is a sub-2-category of the functor 2-category $\mathrm{Adj}(\mathbf{LInd})_{\mathrm{lx}} = [\mathbf{Adj}, \mathbf{LInd}]_{\mathrm{lx}}$, we may study "liftings" as in Definition V.1 quite generally by studying functor 2-categories of the form $[\mathrm{D}, \mathcal{C}]_{\mathrm{lx}}$ for some 2-category D of "diagram shapes".

We shall pair this with the following simple observation about when the codomain fibration restricts to a subcategory. Let $\mathbb{C}$ be any category with a wide subcategory of *tight* maps. Let Lift be the full subcategory of $\mathbb{C}^{\to}$ whose objects are tight maps. Thus, objects of Lift are *liftings*—tight maps $t : C \to C'$—and morphisms are as in $\mathbb{C}^{\to}$.

Lemma VIII.1. *Suppose that for any tight map t the pullback along an arbitrary map f exists and is tight. Then the codomain fibration restricts to a fibration* $\mathrm{cod} : \mathrm{Lift} \to \mathbb{C}$.

If every map is tight this is the codomain fibration. If just the monos are tight, this is the subobject fibration.

In this light, our lifting theorem becomes a statement about the existence of pullbacks in the 2-category $[\mathrm{D}, \mathcal{C}]_{\mathrm{lx}}$. Suppose that $\mathcal{C}$ has a sub-class of *tight* 1-cells which are all fibrations in $\mathcal{C}$—intuitively, the fibrations that strictly preserve structure—and that the underlying 1-category $\mathcal{C}_0$ is such that (1) the pullback of tight 1-cell along arbitrary 1-cells exists in $\mathcal{C}_0$; and (2) these pullbacks are preserved by every $\mathcal{C}_0(C, -) : \mathcal{C}_0 \to \mathbf{Cat}$. A direct calculation using the universal properties similar to that outlined in Section V then shows the following.

Proposition VIII.2. *In the situation just outlined, every componentwise tight transformation $\tau : F \Rightarrow G$ in $[\mathrm{D}, \mathcal{C}]_{\mathrm{lx}}$ has pullbacks along arbitrary transformations, which are componentwise tight.*

Theorem VIII.3. *In the situation of Proposition VIII.2, define a* lifting *to be a componentwise tight lax natural transformation $\tau : \widehat{H} \Rightarrow H : \mathrm{D} \to \mathbb{C}$, and let* Lift *be the full subcategory of the arrow category $([\mathrm{D}, \mathcal{C}]_{\mathrm{lx}})_0^{\to}$ with objects the liftings. Then the codomain functor restricts to a fibration* $\mathrm{Lift} \to [\mathrm{D}, \mathcal{C}]_{\mathrm{lx}}$.

Example VIII.4. We recover Theorem V.2 as follows. Take $\mathrm{D} := \mathbf{Adj}$, $\mathcal{C} := \mathbf{LInd}$, and say a morphism in $\mathbf{LInd}$ is tight if it is a CBPV fibration. Since $\mathbf{CBPV}^{\times}_{\mathrm{lx}}$ is closed under pullbacks of tight maps, the fibration from Theorem VIII.3 restricts to the fibration claimed in Theorem V.2. Corollary V.3 follows by instantiating the theorem in $\mathbf{LInd}^{\mathrm{co}}$.

We obtain a version for CBV by varying the 2-category D.

Write $J : \mathbf{CCCat}^{\mathrm{op}} \to 2\text{-}\mathbf{Cat}$ for the 2-functor sending $\mathbb{V}$ to the 2-category $\mathbb{V}\text{-}\mathbf{Act}_{\mathrm{lx}}$ of (left) $\mathbb{V}$-actions and lax maps (see [48, §3]). $J(f)$ is the 2-functor $\mathbb{W}\text{-}\mathbf{Act}_{\mathrm{lx}} \to \mathbb{V}\text{-}\mathbf{Act}_{\mathrm{lx}}$ precomposing by $f \times \mathrm{id}$, and the action on 2-cells is defined similarly. Applying the 2-Grothendieck construction, we get a 2-category $\mathbf{Act}_{\mathrm{lx}}$ of actions and lax maps. These are the "pre-models" for CBV: indeed, a monad internal to this 2-category (see [49]) is exactly a left action together with a monad that is strong for the action (e.g. [15, §3]).

Example VIII.5. Take $\mathrm{D} := \mathbf{Mnd}$ and $\mathcal{C} := \mathbf{Act}_{\mathrm{lx}}$ as in Remark IV.2 and say a morphism (f, F) in $\mathbf{Act}$ is tight if it is a strict map of actions and both f and F are fibrations which strictly preserve cartesian closed structure. These are fibrations in $\mathbf{Act}_{\mathrm{lx}}$. A lifting then consists of:

- Two triples consisting of a cartesian closed category, an action, and a monad that is strong for the action:
$$(\widehat{\mathbb{V}}, \widehat{\mathbb{V}} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \widehat{T}) \qquad (\mathbb{V}, \mathbb{V} \times \mathbb{C} \to \mathbb{C}, T)$$
- A strict cartesian closed fibration $p : \widehat{\mathbb{V}} \to \mathbb{V}$ and a fibration $P : \widehat{\mathbb{C}} \to \mathbb{C}$ such that (p, P) is a strict map of actions and P strictly preserves the monad.

Any CBV lifting is such a lifting. Moreover, since the sub-2-category of cartesian categories acting on themselves by the product structure is closed under pullbacks, applying Theorem VIII.3 to a lax map of CBV models yields a lifted CBV model, defined by pullback.

IX. Perspectives

In this paper we have shown how fibrations internal to 2-categories can be used to define a mathematically robust and principled definintion of logical relations for CBPV. We have illustrated the viability of our definition by showing how familiar fibrational logical relations can be generalized to CBPV. As our main application, we use our framework to prove, for the first time, a conservativity property of CBPV.

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Appendix A
The basic definitions of 2-category theory

We briefly review the definitions of 2-categories, 2-functors, transformations, and modifications. For reasons of space we omit the coherence axioms: for full details seeerr e.g. [21], [22].

Definition A.1. A *2-category* $\mathcal{C}$ consists of:

1) A collection of objects $A, B, \ldots$.
2) For all objects A and B, a collection of morphisms from A to B, which are themselves related by morphisms: thus we have a *hom-category* $\mathcal{C}(A,B)$ whose the objects $f,g : A \to B$ are *morphisms* (or *1-cells*) and whose morphisms are *2-cells* $\sigma, \tau : f \Rightarrow g$.
3) For all A, B and C a *composition* functor $\circ_{A,B,C}$: $\mathcal{C}(B,\mathcal{C}) \times \mathcal{C}(A,B) \to \mathcal{C}(A,C)$ and, for all A an *identity* 1-cell $\mathrm{id}_A : A \to A$, such that composition is associative and unital on both 1-cells and 2-cells.

The hom-category structure means the following. For any 1-cell $f : A \to B$ there is an identity 2-cell $\mathrm{id}_f : f \Rightarrow f$. Moreover, given $\sigma : f \Rightarrow g : A \to B$ and $\tau : g \Rightarrow h : A \to B$ we can *vertically compose* to obtain a 2-cell $\tau * \sigma : f \Rightarrow h$. The functoriality of composition says that, given $\sigma : f \Rightarrow f' : A \to \mathcal{B}$ and $\tau : g \Rightarrow g' : B \to C$ we can *horizontally compose* to obtain a 2-cell $\tau \circ \sigma : g \circ f \Rightarrow g' \circ f'$, and that the two composition operations are related by the so-called interchange law. The names correspond to how the operations look when drawn in $\mathbf{Cat}$ (see [25, §II.5]). Following standard 2-categorical practice, we sometimes write simply $A\sigma$ or $f\sigma$ instead of $\mathrm{id}_A \circ \sigma$ or $\mathrm{id}_f \circ \sigma$, and similarly for composition on the other side.

Every 2-category $\mathcal{C}$ has three duals, corresponding to reversing just the 1-cells, just the 2-cells, or both. $\mathcal{C}^{\mathrm{op}}$ has $\mathcal{C}^{\mathrm{op}}(A,B) := \mathcal{C}(B,A)$, so just the 1-cells are reversed. $\mathcal{C}^{\mathrm{co}}$ has $\mathcal{C}^{\mathrm{co}}(A,B) := \mathcal{C}(A,B)^{\mathrm{op}}$, so just the 2-cells are reversed. $\mathcal{C}^{\mathrm{coop}}$ has $\mathcal{C}^{\mathrm{co}}(A,B) := \mathcal{C}(B,A)^{\mathrm{op}}$, so both 1-cells and 2-cells are reversed.

As indicated above, the prototypical example is $\mathbf{Cat}$: the objects are functors, the 1-cells are functors, and the 2-cells are natural transformations.

Definition A.2. A *2-functor* $F : \mathcal{C} \to \mathcal{D}$ is a mapping on objects, 1-cells, and 2-cells which preserves both horizontal and vertical composition, so that $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_A) = \mathrm{id}_{FA}$ on 1-cells, and $F(\tau \circ \sigma) = F(\tau) \circ F(\sigma)$, $F(\tau * \sigma) = F(\tau) * F(\sigma)$, and $F(\mathrm{id}_f) = \mathrm{id}_{Ff}$ on 2-cells.

When considering functors between monoidal categories, there are four grades of strictness we can ask for: strict, pseudo (strong), lax, or oplax. The same applies for morphisms between 2-functors.

Definition A.3. Let $F, G : \mathcal{C} \to \mathcal{D}$ be 2-functors. A *lax natural transformation* $\sigma : F \to G$ consists of a 1-cell $\sigma_C : FC \to$

GC for each $C \in \mathcal{C}$ together with, for every 1-cell $f : C \to C'$ in $\mathcal{C}$, a 2-cell σ_f witnessing the naturality as shown below:

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FC' \\ {\scriptstyle \sigma_C}\downarrow & \overset{\sigma_f}{\swArrow} & \downarrow{\scriptstyle \sigma_{C'}} \\ GC & \xrightarrow[Gf]{} & GC' \end{array}$$

This is required to satisfy unit and associativity laws, and be natural in f. An *oplax* natural transformation is a transformation in $\mathcal{C}^{\mathrm{co}}$: the 2-cell σ_f is reversed. A *pseudonatural* transformation is a transformation in which every σ_f is an isomorphism. A *strict* (or *2-natural*) transformation is one in which every σ_f is the identity.

Transformations play the same role as natural transformations in category theory. Since 2-category theory has an extra layer of data, there is an extra form of morphism. We state the definition for lax natural transformations; corresponding definitions hold for oplax, pseudo, and strict transformations.

Definition A.4. Let $\sigma, \tau : F \to G : \mathcal{C} \to \mathcal{D}$ be lax natural transformations. A *modification* $\Gamma : \sigma \to \tau$ consists of a 2-cell $\Gamma_C : \sigma_C \Rightarrow \tau_C$ for each $C \in \mathcal{C}$, subject to an axiom making it compatible with the 2-cells σ_f and τ_f.

For any 2-categories $\mathcal{C}$ and $\mathcal{D}$, we therefore obtain four 2-functor categories. For each $w \in \{\mathrm{st}, \mathrm{ps}, \mathrm{lx}, \mathrm{oplx}\}$ there is a 2-category $[\mathcal{C}, \mathcal{D}]_w$ with objects the 2-functors $\mathcal{C} \to \mathcal{D}$, 1-cells either the strict, pseudo, lax, or oplax transformations, and 2-cells the modifications.